Queens that attack at most one other, or exactly one other

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Abstract

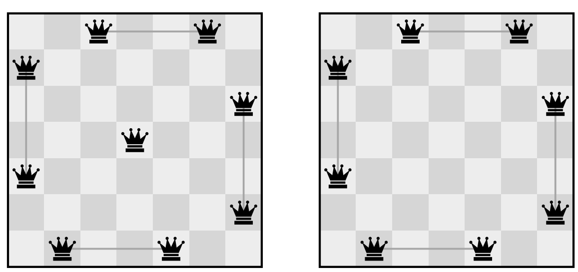
Let $q_1(n)$ be the maximum number of queens that can be placed on an $n \times n$ chessboard so that each queen attacks at most one other, and let $Q_1(n)$ be the corresponding maximum when each queen attacks exactly one other. Counting arguments give $q_1(n) \leq \lfloor 4n/3 \rfloor$ and $Q_1(n) \leq 2\lfloor 2n/3 \rfloor$. We prove that both are attained for every $n \geq 6$:

$$q_1(n) \;=\; \lfloor 4n/3 \rfloor, \qquad Q_1(n) \;=\; 2\lfloor 2n/3 \rfloor.$$

In particular $q_1(n) = Q_1(n)$ unless $n \equiv 1 \pmod 3$, in which case $q_1(n) = Q_1(n) + 1$. Makay had established the q_1 bound for all $n \geq 354$; the argument here covers all $n \geq 6$, and Q_1 in the same stroke.

1 The problem

Two queens attack one another when they share a row, a column, or a diagonal, with intervening pieces blocking in the usual way. Write $q_1(n)$ for the maximum number of queens that can be placed on an $n \times n$ board so that each queen attacks *at most* one other queen, and write $Q_1(n)$ for the maximum when each queen attacks *exactly* one other. Figure 1 shows an optimal placement of each kind on the 7×7 board. Both problems, and the broader family in which each queen attacks exactly k others, are discussed by Gardner [2].



$q_1(7) = 9$ $\qquad$ $Q_1(7) = 8$

Figure 1: Optimal placements on the 7×7 board. The left-hand board realises $q_1(7) = 9$ and the right-hand board realises $Q_1(7) = 8$. Both are optimal because they achieve the bounds derived below.

Both quantities admit elementary upper bounds by a line-budget count [3, 5]. In any such configuration the queens partition into p attacking pairs and s isolates ($s = 0$ in the exact-one case). Each attacking pair occupies at least three distinct rows or columns among the n rows and n columns—a row-pair uses one row and two columns, a column-pair uses two rows and one column, a diagonal pair uses two of each—and those rows and columns are then forbidden to every other queen. An isolated queen occupies one row and one column exclusively. Hence

$$3p + 2s \ \le\ 2n, \qquad \text{number of queens} \ =\ 2p + s \ =\ \tfrac{2}{3}(3p + 2s) - \tfrac{1}{3}s \ \le\ \tfrac{4n}{3}.$$

In particular

$$q_1(n) \ \le\ \left\lfloor \frac{4n}{3} \right\rfloor, \tag{1}$$

and, taking $s = 0$ and noting that the count is even,

$$Q_1(n) \ \le\ 2\left\lfloor \frac{2n}{3} \right\rfloor. \tag{2}$$

When $n \not\equiv 1 \pmod 3$ the two bounds coincide, so an optimal at-most-one placement is necessarily exact-one. When $n \equiv 1 \pmod 3$ the at-most-one bound is larger by one, and any optimal q_1-placement consists of $\lfloor 2n/3 \rfloor$ attacking pairs together with a single isolate.

The sequences $q_1(n)$ and $Q_1(n)$ are recorded as A260113 and A051754 in the OEIS [4]. Direct computation for small n shows that (1) fails at $n = 3, 4, 5$ and that (2) fails at $n = 3, 5$. From $n = 6$ onward, every computed value matches the bound. The pattern suggested by those tables is

$$\begin{aligned} q_1(n) &= \lfloor 4n/3 \rfloor \quad \text{for all } n \ge 6, \\ Q_1(n) &= 2\lfloor 2n/3 \rfloor \quad \text{for all } n \ge 6, \end{aligned} \tag{3}$$

and, more finely, $q_1(n) = Q_1(n)$ unless $n \equiv 1 \pmod 3$, in which case $q_1(n) = Q_1(n) + 1$.

The rest of this note consists of a proof of this result:

Theorem 1. *For every $n \ge 6$,*

$$q_1(n) \ =\ \lfloor 4n/3 \rfloor, \qquad Q_1(n) \ =\ 2\lfloor 2n/3 \rfloor.$$

Sections 2–4 give a uniform construction for every $n \ge 216$. Sections 5–6 cover $6 \le n \le 215$ by the same paste on smaller bases, together with a handful of stored boards.

Makay [1] proved that $q_1(n) = \lfloor 4n/3 \rfloor$ for all $n \ge 354$. The construction forms the product of an explicit 6×6 configuration of eight queens, each attacking exactly one other, with a solution T_s to the classic s-queens problem on the torus (a placement of s queens so that no queen attacks any other, even when rows, columns, and diagonals wrap around). That product produces infinitely many bound-optimal boards with a wide empty central channel, which Makay then expands by a collection of extension tables and gadgets. The present note is an attempt to simplify and extend that approach. The product strategy is retained, and with it the wide central channel that makes it possible to insert queens in a corner of the board without introducing conflicting attacks. Makay's ad hoc gadgets are replaced by a single recursive step: paste a bound-optimal $m \times m$ board into the bottom-right corner of a channelled $B \times B$ board. The construction works for all "large enough" n, which in this case means $n \ge 216$.

The construction for $n \ge 216$ relies on a finite set of q_1-optimal placements for $6 \le n \le 29$ drawn in Appendix A. All of them occupy a relatively narrow diagonal band (small β, defined below), so they paste onto the product boards mentioned above. A similar approach works for many, but not all, sizes $n \le 215$; the remaining cases are handled by additional placements of sizes 12, 15, 18, and 24 chosen to have relatively wide queen-free diagonal channels, together with a few other special cases. These placements were found by SAT/CP-SAT search subject to a band or channel constraint.

2 The product construction and the central channel

The key to both Makay's argument and the present one is the product of a specific 6×6 configuration with a solution to the classic s-queens problem on the torus (no queen attacking any other, even with wrap-around). For each integer s coprime to 6 write T_s for the placement

$$T_s \;=\; \bigl\{\, (i,\, 2i \bmod s) : 0 \le i < s \,\bigr\}. \tag{4}$$

Placing s queens at $(i, \sigma(i))$ solves the classic s-queens problem on the torus: no two share a row, a column, or a wrap-around diagonal. Equivalently, the values $\sigma(i)$, the values $i - \sigma(i)$, and the values $i + \sigma(i)$ are all distinct modulo s. (The same condition is often called a modular s-queens solution.) Such placements exist if and only if $\gcd(s, 6) = 1$ [6].

Lemma 2 (Non-attacking queens on the torus)**.** *The placement T_s solves the classic s-queens problem on the torus if and only if* $\gcd(s, 6) = 1$.

Proof. Queens at $(i, 2i \bmod s)$ occupy every row. They occupy every column if and only if $i \mapsto 2i$ is bijective modulo s, i.e. s is odd. The wrap-around main-diagonal indices are $i - 2i \equiv -i$, which run through all residues once s is odd. The wrap-around anti-diagonal indices are $i + 2i \equiv 3i$, which run through all residues if and only if $\gcd(3, s) = 1$. Combining the two coprimality conditions gives $\gcd(s, 6) = 1$. □

In particular one may take $s = 6t \pm 1$ for any integer $t \ge 1$.

Two measures of diagonal "width" will be important in the arguments that follow: for a placement S write $d(S) = \min\{|r - c| : (r, c) \in S\}$ and $W(S) = \max\{|r - c| : (r, c) \in S\}$, and

$$\chi(S) \;=\; \max\bigl\{0,\; 2d(S) - 1\bigr\}, \qquad \beta(S) \;=\; 2W(S) + 1 \tag{5}$$

for the width of the central *channel* of S (the empty diagonals with $|r - c| < (\chi + 1)/2$, or none if a queen already sits on the main diagonal) and the width of the smallest central *band* that covers every queen. Both widths are odd once the channel is nonempty.

The seed configuration is the eight-queen placement on a 6×6 board shown in Figure 2,

$$S_6 \;=\; \bigl\{(0,1),\, (0,2),\, (1,5),\, (2,5),\, (3,0),\, (4,0),\, (5,3),\, (5,4)\bigr\}. \tag{6}$$

Each queen attacks exactly one other — two row-pairs and two column-pairs — so $|S_6| = 8 = q_1(6) = Q_1(6)$. No two queens share a diagonal, and the central channel has width 1: $\chi(S_6) = 1$.

Now view a $ks \times ks$ board as a $k \times k$ array of $s \times s$ blocks, and insert a copy of T_s in each block whose corresponding square belongs to a configuration S:

$$S \otimes T_s \;=\; \bigl\{\, (sr + i,\; sc + (2i \bmod s)) : (r, c) \in S,\; 0 \le i < s \,\bigr\}. \tag{7}$$

The fact that T_s is a *toroidal* s-queens solution is what makes the product safe. A diagonal that stays inside a block is an ordinary diagonal of T_s. A diagonal that leaves a block re-enters a neighbouring block at the opposite side, which is exactly a wrap-around diagonal of T_s. Both are empty. The smallest product is $S_6 \otimes T_5$, a 30×30 board with a channel of width 5. Figure 3 draws the next size in the family $36t - 6$, namely $S_6 \otimes T_{11}$ ($t = 2$): each occupied cell of S_6 becomes an 11×11 block carrying T_{11}, and the channel of width 1 in S_6 opens into a channel of width 11. That channel is wide enough to receive a bound-optimal 8×8 corner (band of width 9), which is the construction "$S_6 \otimes T_{11} + 8$" drawn in Figure 4.

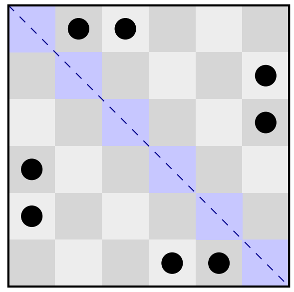


Figure 2: The seed configuration S_6. The shaded main diagonal is the channel of width 1; every neighbouring diagonal already meets a queen.

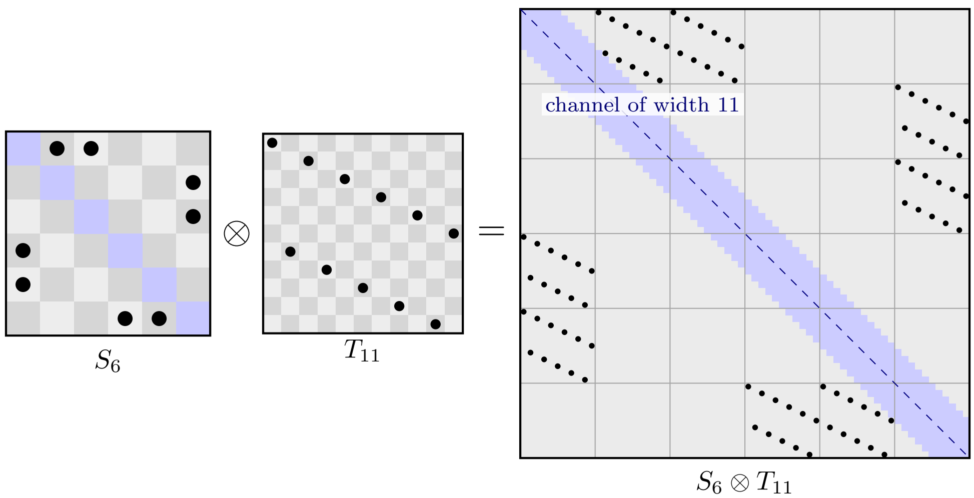


Figure 3: The tensor product $S_6 \otimes T_{11}$ ($t = 2$). Each occupied cell of S_6 is replaced by a copy of T_{11}. The 6×6 block grid is overlaid on the 66×66 board. All queens lie outside the shaded channel of width 11.

Lemma 3 (Product)**.** *Let S be a bound-optimal Q_1-placement on a $k \times k$ board with $3 \mid k$, and let $\gcd(s, 6) = 1$. Then $S \otimes T_s$ is a bound-optimal Q_1-placement on the $(ks) \times (ks)$ board. In particular $S_6 \otimes T_s$ realises*

$$q_1(6s) \;=\; Q_1(6s) \;=\; \frac{4}{3} \cdot 6s.$$

Proof. There are $|S|\, s = (4k/3)s = 4N/3$ queens on the $N = ks$ board, so it is enough that each queen attack exactly one other.

View the board as a $k \times k$ array of $s \times s$ blocks, with a copy of T_s in each block belonging to S. Because $3 \mid k$ and S meets the Q_1 bound, there are $p = 2k/3$ pairs and $3p = 2k$ occupied rows and columns, so the leftover line-budget is zero. Every pair of S is therefore axis-aligned, and no two queens of S share a diagonal.

Rows (resp. columns) of distinct blocks are disjoint. Within a single coarse row the copies occupy the same fine row i of each occupied block, so a row-attack occurs if and only if S itself has a row-pair, copied once at each fine index. At the Q_1 bound a coarse row holds at most one such pair, and each of its queens has exactly one partner. Column attacks are symmetric.

No new diagonal attacks can occur inside a block, since T_s is an ordinary s-queens solution. There are none across blocks either, since T_s is a *toroidal* s-queens solution: a Euclidean diagonal that leaves a block re-enters a neighbour at the opposite side, which is a wrap-around of T_s. The only surviving attacks are therefore the s parallel copies of each axis-aligned pair of S. □

The same product creates a wide empty channel about the main diagonal. The diagonal coordinate of a queen $(sr + i,\ sc + (2i \bmod s))$ splits as coarse plus fine,

$$s(r - c) + \big(i - (2i \bmod s)\big). \tag{8}$$

The configuration S contributes a multiple of s, and T_s contributes its own diagonal index.

Lemma 4 (Channel)**.** *Let S be a placement with a channel of width $\chi \geq 1$, and let $\gcd(s, 6) = 1$. Then*

$$\chi\big(S \otimes T_s\big) \ = \ s \cdot \chi(S).$$

In particular $\chi(S_6 \otimes T_s) = s$: the boards $S_6 \otimes T_{6t-1}$ of size $36t - 6$ have a channel of width $6t - 1$, and the boards $S_6 \otimes T_{6t+1}$ of size $36t + 6$ have a channel of width $6t + 1$.

Proof. Write $f(i) = i-(2i \bmod s)$ for the integer (not merely modular) diagonal offset of T_s. For odd s the map f is a bijection from $\{0, \ldots, s-1\}$ onto the balanced interval $\big[-(s-1)/2,\ (s-1)/2\big]$. This lift is special to $\sigma(i) = 2i \bmod s$: a general modular s-queens permutation need not realise its wrap-around diagonal indices as that interval of ordinary integers.

Combined with (8), each coarse difference $r - c = \kappa$ therefore expands to the s consecutive fine diagonals $s\kappa + \big[-(s-1)/2,\ (s-1)/2\big]$. Adjacent coarse indices abut: the largest value for κ is one less than the smallest value for $\kappa + 1$. The χ empty coarse diagonals about 0 therefore become $s\chi$ empty fine diagonals about 0.

As a numeric check: S_6 has $\min|r - c| = 1$, so $\chi(S_6) = 1$. For $s = 11$ the values $f(i)$ fill $[-5, 5]$, and $S_6 \otimes T_{11}$ has $\min|r - c| = 6$ with empty diagonals $r - c \in \{-5, \ldots, 5\}$, hence $\chi = 11$. □

The channel of the product therefore has width $s\chi$; no queen occupies it, and in particular the main diagonal is empty.

Makay's proof uses precisely this channel. The large empty region about the diagonal is a blank canvas on which extension tables and gadgets can be drawn, pushing the construction from the sparse set of product sizes $\{36t \pm 6\}$ up to every sufficiently large n. The construction below keeps the channel and discards the catalogue of gadgets. Instead, the approach is to paste an already-solved smaller board into a corner, using the channel solely as a guarantee that the two pieces do not attack one another along a main diagonal.

3 Pasting into a corner

Let P be a bound-optimal Q_1-placement of size B with $3 \mid B$ and a channel of width $\chi(P) \geq 1$, and let M be a q_1-placement of size m whose queens occupy a band of width $\beta(M)$. The paste condition $\beta(M) \leq \chi(P)$ of the next lemma is exactly $W(M) \leq d(P) - 1$: the band of M fits inside the channel of P. Form an $(B + m) \times (B + m)$ board by placing P in the top-left $B \times B$ block and M in the bottom-right $m \times m$ block, as in Figure 4: every queen of P lies outside

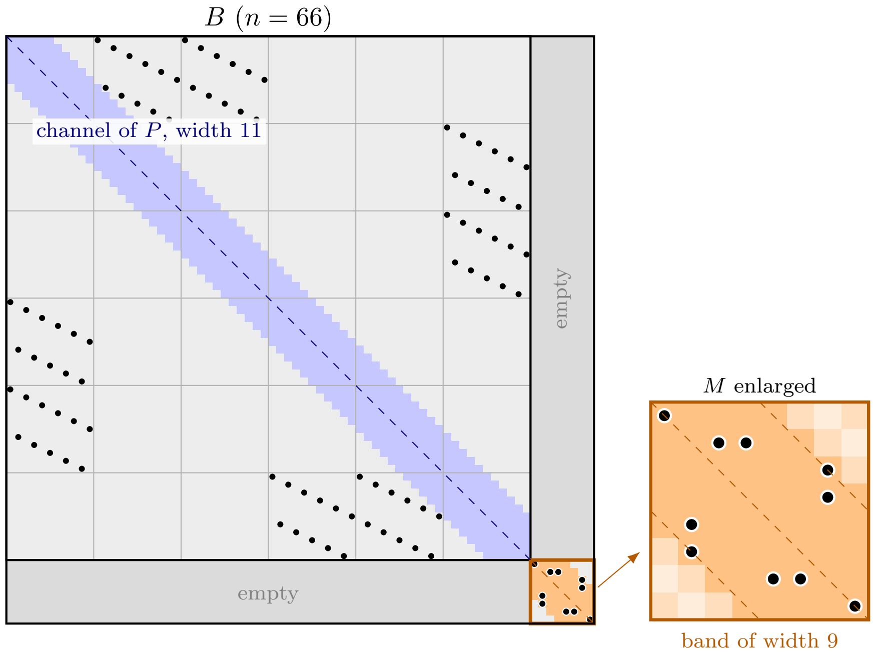


Figure 4: Corner paste of a bound-optimal 8×8 board M ($q_1(8) = 10$) onto $S_6 \otimes T_{11}$ from Figure 3. Every queen of $S_6 \otimes T_{11}$ lies outside the blue channel of width 11; every queen of M lies on the orange band of width 9 (enlarged at right). The L-shaped strip of new cells is empty. The band of M fits in the channel of P (here $9 < 11$; the criterion of Lemma 5 is $\beta \leq \chi$, and equality is allowed), so the paste $S_6 \otimes T_{11} + 8$ is a bound-optimal q_1-placement on the 74×74 board.

the blue channel, every queen of M lies on the orange band, and the L-shaped strip of new cells between the two blocks is empty.

Lemma 5 (Corner paste)**.** *If $\beta(M) \leq \chi(P)$, then the pasted placement is a legal q_1-configuration on the $(B+m) \times (B+m)$ board. If in addition M is bound-optimal for q_1 and $m = 2$ or $m \geq 6$, then the pasted placement is bound-optimal:*

$$q_1(B+m) \; = \; q_1(B) + q_1(m) \; = \; \left\lfloor \frac{4(B+m)}{3} \right\rfloor.$$

The same statement holds with Q_1 in place of q_1 whenever M itself is a Q_1-configuration (in particular, whenever $m \not\equiv 1 \pmod 3$).

Proof. Rows and columns of the two blocks are disjoint, so no new row or column attack is created. Anti-diagonals of the top-left block meet squares (x, y) with $x + y \leq 2B - 2$, while anti-diagonals of the bottom-right block meet squares with $x + y \geq 2B$; the two ranges are disjoint, so no new anti-diagonal attack is created either. The only possible new attacks are therefore along main diagonals $x - y = \text{const}$.

A main diagonal that meets both blocks must pass through the $B \times B$ channel region in between. Equivalently: if $(r, c) \in P$ and $(B+i, B+j) \in M$ lie on the same main diagonal, then $r - c = i - j$. The right-hand side is one of the $\beta(M)$ band indices of M, while the left-hand side lies outside the $\chi(P)$ empty channel indices of P. The inequality $\beta(M) \le \chi(P)$ — equivalently $W(M) \le d(P) - 1$ — makes those two sets of indices disjoint — the band of M fits inside the channel of P — so no such pair exists.

No existing attack is blocked either: the open segment between any two queens of P (resp. M) lies in the bounding box of those two queens, hence inside that block. The pasted attack graph is therefore the disjoint union of the two graphs and is a legal q_1-configuration, and a legal Q_1-configuration if both pieces are.

For the count: $q_1(B) = 4B/3$ because $3 \mid B$ and B is bound-optimal, while $q_1(m) = \lfloor 4m/3 \rfloor$ for $m = 2$ and for every $m \ge 6$. Adding these and using $3 \mid B$ gives $\lfloor 4(B+m)/3 \rfloor$. The same arithmetic applies to Q_1 upon replacing $\lfloor 4m/3 \rfloor$ by $2\lfloor 2m/3 \rfloor$, which agrees with $q_1(m)$ except when $m \equiv 1 \pmod 3$. □

Size 2 realises $q_1(2) = 2$ by the placement $\{(0,0),(1,1)\}$, occupying a single diagonal ($\beta = 1$), and is recorded in Table 1; the catalog uses it freely (for instance $104 = S_6 \otimes T_{17} + 2$). Size 4 misses the q_1 bound — $q_1(4) = 4 < \lfloor 16/3 \rfloor = 5$ — and is never used as a corner.

4 The bound for large n

The two product families of Lemma 4 are $S_6 \otimes T_{6t-1}$ of size $36t-6$ (channel of width $6t-1$) and $S_6 \otimes T_{6t+1}$ of size $36t+6$ (channel of width $6t+1$). The band widths of bound-optimal q_1-placements with $6 \le m \le 29$ are recorded in Table 1; the worst entries on this range are $\beta = 37$. For $t \ge 6$ the plus family has channel $6t+1 \ge 37$, so every such corner can be pasted onto $S_6 \otimes T_{6t+1}$. That window

$$\big[36t+6+6,\, 36t+6+29\big] \;=\; \big[36t+12,\, 36t+35\big]$$

covers twenty-four consecutive sizes. The twelve preceding sizes $\big[36t,\, 36t+11\big]$ are pastes of $m = 6,\ldots,17$ onto the minus family $S_6 \otimes T_{6t-1}$. On that short range the worst band is $\beta(17) = 27$, inside the minus channel $6t-1 \ge 35$. The two windows therefore abut for $t \ge 6$, starting at $n = 216$. No residue is missing. This proves:

Proposition 6. *$q_1(n) = \lfloor 4n/3 \rfloor$ for every $n \ge 216$.*

Corollary 7. *$Q_1(n) = 2\lfloor 2n/3 \rfloor$ for every $n \ge 216$.*

Proof. When $n \not\equiv 1 \pmod 3$ the two bounds coincide. When $n \equiv 1 \pmod 3$, a bound-optimal q_1-placement consists of $\lfloor 2n/3 \rfloor$ attacking pairs and a single isolate. At the bound, the leftover line-budget is 0, so every pair is axis-aligned and the isolate occupies its own row, column, and both diagonals. Deleting it cannot create an attack, so the remainder is a bound-optimal Q_1-placement. □

The only input to this argument that is not a direct inspection of S_6 or an application of Lemmas 2–5 is the range $6 \le m \le 29$ of Table 1. Each of those rows is witnessed by an explicit q_1-optimal placement, drawn in Appendix A.

Table 1: Band width β of a bound-optimal q_1-placement, for the corners used in the large-n argument. The worst entries on $6 \le m \le 29$ are $\beta = 37$, so each such corner pastes onto the plus family at $t = 6$ (channel of width 37). Size 2 is the trivial corner $\{(0,0),(1,1)\}$.

m	β	m	β	m	β	m	β
2	1						
6	9	12	17	18	29	24	37
7	11	13	19	19	29	25	33
8	9	14	17	20	31	26	37
9	13	15	21	21	37	27	37
10	13	16	25	22	29	28	37
11	13	17	27	23	33	29	37

5 Many sizes below the threshold

Proposition 6 is stated from $n = 216$ because that is the point at which the two product families $36t \pm 6$, with corners $6 \le m \le 29$, cover every residue modulo 36. The paste of Lemma 5, however, can still be applied for $t < 6$ if we are willing to leave some gaps in the sizes that are covered. Every smaller product $S_6 \otimes T_s$, and every tensor of a channelled configuration with some T_s, is an equally legal base; applying the same corners against those smaller bases covers the great majority of n in $6 \le n \le 215$. Appendix B records an explicit recipe for each such n.

Concretely, $S_6 \otimes T_s$ already supplies channelled bases of sizes

30, 42, 66, 78, 102, 114, 138, 150, 174, 186, 210

(and the companion sequence $S_6 \otimes T_{6t+1}$ of size $36t + 6$). Each such base B covers every m with $6 \le m \le 29$ and $\beta(m) \le \chi(B)$, hence a run of sizes $B + m$ whose length grows with t. The first few bases leave gaps; by $S_6 \otimes T_{17}$ (size 102, channel of width 17) the gaps inside each window of thirty-six have largely closed, and the remaining holes are sparse.

A handful of additional bases come from Lemma 4 applied to bound-optimal boards that happen to have a strictly positive channel. Write C_n for such a placement (not necessarily the narrow corner of Table 1): the configuration S_6 is the prototype ($\chi = 1$), and four further q_1-optimal placements of size at most 24 are used as tensor factors, namely C_{12}, C_{15}, C_{18}, C_{24}. Tensoring these with T_s produces, among others, $C_{12} \otimes T_5 = 60$, $C_{12} \otimes T_7 = 84$, $C_{15} \otimes T_7 = 105$, $C_{24} \otimes T_5 = 120$, $C_{12} \otimes T_{11} = 132$, which fill several of the gaps that the six-square family leaves. Wider channelled boards of sizes 27, 30, 33, and 36 exist, but every tensor they produce below $n = 216$ is already a paste onto S_6 or C_{24} — for instance $135 = C_{24} \otimes T_5 + 15$ in place of $C_{27} \otimes T_5$ — so those four boards are omitted.

The four tensor factors are recorded in Table 2 and drawn in Appendix C. Three further stored boards with $n > 41$ appear as leftover C_n below $n = 216$: they fill the gap $(42, 66)$ recorded in Table 3. Tensors of C_{45} and C_{48} begin at $225 = C_{45} \otimes T_5$, already in the large-n range, so they are omitted. The large-n argument of the previous section does not use them.

The Table 1 corner $m = 10$ ($\beta = 13$) is an ordinary paste, not a leftover primitive. It produces

$$88 = S_6 \otimes T_{13} + 10, \qquad 124 = S_6 \otimes T_{19} + 10, \qquad 130 = C_{24} \otimes T_5 + 10, \qquad 160 = S_6 \otimes T_{25} + 10$$

Table 2: Channelled q_1-optimal configurations C_n of size at most 24 used as tensor factors below $n = 216$. The column β is the band width of the *narrow* corner from Table 1; the column χ is the channel width of C_n, a (possibly different) placement of the same size. Drawings and coordinates are in Appendix C.

n	β (narrow)	χ (channel)	typical tensors
6	9	1	$S_6 \otimes T_s$
12	17	3	$C_{12} \otimes T_5,\ T_7,\ T_{11}$
15	21	3	$C_{15} \otimes T_5,\ T_7,\ T_{13}$
18	29	3	$C_{18} \otimes T_5,\ T_7,\ T_{11}$
24	37	5	$C_{24} \otimes T_5,\ T_7$

Table 3: Gap boards for $(42, 66)$. The three C_n are leftover channelled q_1-optimal placements (hence Q_1-optimal, since $n \not\equiv 1 \pmod 3$). Size $60 = C_{12} \otimes T_5$ already sits in Table 2; sizes 51–53 come from C_{45}, and sizes 54–59 are pastes onto C_{48}. Drawings and coordinates are in Appendix C.

n	queens	χ	sizes it buys
45	60	9	45–47; pastes $45 + 6, 8$
48	64	13	48–50; pastes $48 + 6, \ldots, 11$
63	84	1	63–65

on channels of width 13, 19, 25, and 35. (The catalog writes +10 rather than isolate on a 9-paste because every bound-optimal $Q_1(9)$ meets $r = c$, a direct check of $Q_1(9) = 12$; the pastes $87 = S_6 \otimes T_{13} + 9$ and its companions therefore occupy the main diagonal.) From $n = 67$ through 215 every remaining size is a product, a tensor, a paste of a Table 1 corner (including $m = 2$ and $m = 10$), or one step of the isolate gadget below. The only stored boards with $n > 41$ sit in the product gap $(42, 66)$ of Table 3.

6 The remaining cases

The isolate gadget is the default tool for $n \equiv 1 \pmod 3$ when no $\equiv 1$ corner fits in the channel (for instance $43 = 42 + 1$, because $\beta(7) = 11 > 7 = \chi(42)$). The only leftover primitives are the three boards of Table 3.

6.1 The isolate gadget

Suppose $N \equiv 0 \pmod 3$ and Q is a bound-optimal Q_1-placement on an $N \times N$ board whose main diagonal is empty. Form an $(N + 1) \times (N + 1)$ board by embedding Q in the top-left $N \times N$ block and placing a single additional queen at (N, N). The new queen shares no row or column with Q. It shares no anti-diagonal with Q either: the anti-diagonal through (N, N) is $r + c = 2N$, which does not meet the smaller board at all ($r + c \leq 2N - 2$ there). It shares the main diagonal $r - c = 0$ with the smaller board, but that diagonal is empty by hypothesis.

Hence the new queen is isolated, and the placement realises

$$q_1(N+1) \;=\; Q_1(N)+1 \;=\; \left\lfloor \frac{4(N+1)}{3} \right\rfloor.$$

Deleting the isolate recovers $Q_1(N+1) = Q_1(N)$: the new queen occupies a private row, column, and both diagonals, so removing it cannot create an attack.

Every product and tensor base constructed above has a strictly positive channel, hence an empty main diagonal, so the gadget applies to each of them. Some pastes remain diagonal-free as well — the corner M need not meet $r = c$ — and the gadget applies to those too; for instance $S_6 \otimes T_{23} + 15$ does not meet $r = c$, so the gadget produces a bound-optimal 154-board (the catalog of Appendix B records the ordinary paste $C_{12} \otimes T_{11} + 22$ for this size). In particular the gadget produces $q_1(n)$ at every $n \equiv 1 \pmod 3$ of the form $B + 1$ with B a channelled base or a diagonal-free paste, including $n = 61$ from $C_{12} \otimes T_5$ and $n = 106$ from $C_{15} \otimes T_7$.

6.2 The gap $(42, 66)$

The three channelled boards of Table 3 are leftover primitives: each is a bound-optimal q_1-placement with a strictly positive channel (and Q_1-optimal, since $n \not\equiv 1 \pmod 3$), and none is a product $S_6 \otimes T_s$, a tensor of a Table 2 seed, or a paste of a Table 1 corner onto such a base. The wide board C_{45} ($\chi = 9$) additionally receives the corners $m = 6$ and $m = 8$, covering 51 and 53; the 6-corner is diagonal-free, so the isolate gadget produces 52. The wide board C_{48} ($\chi = 13$) additionally receives the corners $m = 2$ and $6 \le m \le 11$, covering 50 and 54–59. These sizes, together with $n = 9$ (which is among the $n \le 41$ data), complete the covering of $[6, 215]$.

6.3 Summary of the covering

Every $n \ge 6$ is obtained in one of the following ways, in decreasing order of typicality:

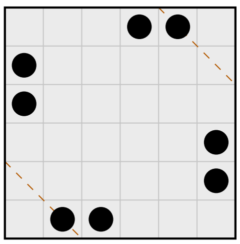
1. $n \ge 216$: a corner paste of some $m \in \{6, \ldots, 29\}$ onto a product board of size $36t - 6$ or $36t + 6$ with $t \ge 6$.
2. a product $S_6 \otimes T_s$ or a tensor $C_k \otimes T_s$ of a Table 2 seed, or a corner paste of a Table 1 board (including $m = 2$ and $m = 10$) onto such a base;
3. the isolate gadget on a channelled base or a diagonal-free paste;
4. a stored board: Table 1 for $6 \le n \le 29$, the twelve boards $30 \le n \le 41$ of Appendix A, or one of the three leftover C_n of Table 3 in the product gap $(42, 66)$.

An explicit recipe for each $n \in [6, 215]$ is Appendix B. The finite geometric data are the q_1-optimal boards of size at most 41 in Appendix A, together with the four wide-channel tensor factors of Table 2 and the leftover boards of Table 3, drawn in Appendix C. Everything else is an algebraic combination of those boards with S_6 and T_s.

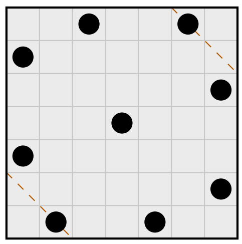
This completes the proof of Theorem 1.

A The corners of Table 1

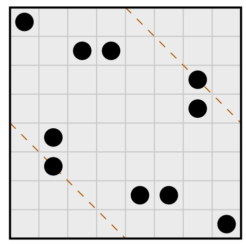
The boards below are the bound-optimal q_1-placements for $6 \le m \le 41$. Those with $6 \le m \le 29$ are the corners of Table 1; the twelve boards $30 \le m \le 41$ are stored data for those sizes only. The two dashed lines on each board are the main-diagonals $r - c = \pm W$ through the outermost queens, where $W = \max|r - c|$ and $\beta = 2W + 1$. No queen lies outside those lines. The $m = 6$ board is the vertical reflection of the seed S_6 in Figure 2; the two realise the same band. The $m = 7$ board is the left-hand placement of Figure 1. Coordinates, with $(0, 0)$ at the top-left, follow the drawings.



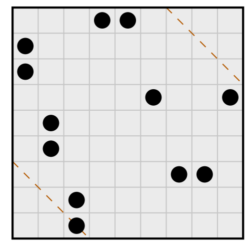
$m = 6$, $\beta = 9$



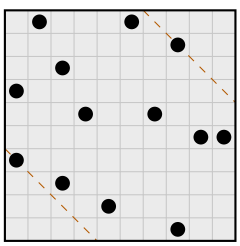
$m = 7$, $\beta = 11$



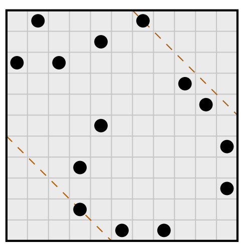
$m = 8$, $\beta = 9$



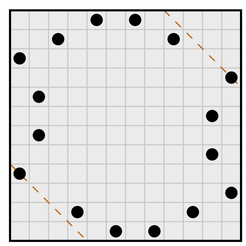
$m = 9$, $\beta = 13$



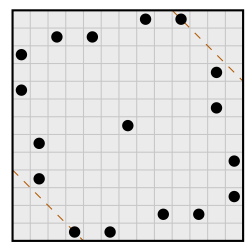
$m = 10$, $\beta = 13$



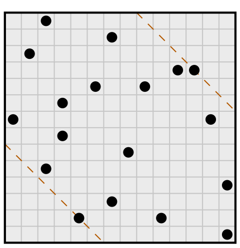
$m = 11$, $\beta = 13$



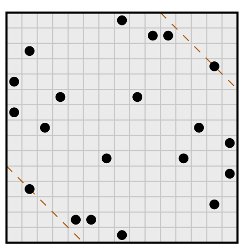
$m = 12$, $\beta = 17$



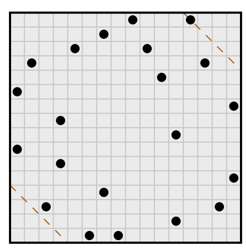
$m = 13$, $\beta = 19$



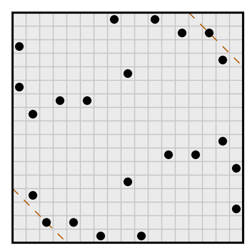
$m = 14$, $\beta = 17$



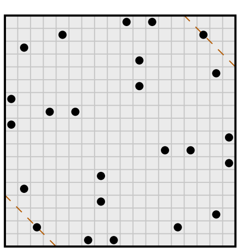
$m = 15$, $\beta = 21$



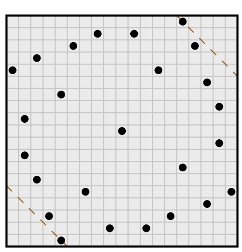
$m = 16$, $\beta = 25$



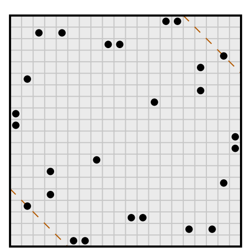
$m = 17$, $\beta = 27$



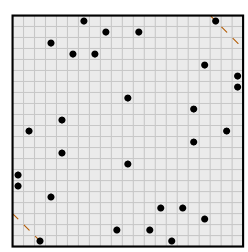
$m = 18$, $\beta = 29$



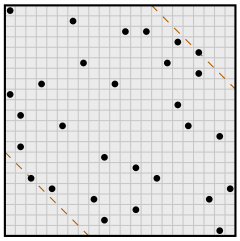
$m = 19$, $\beta = 29$



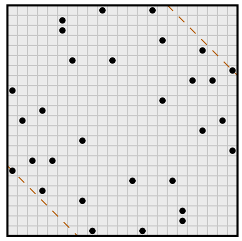
$m = 20$, $\beta = 31$



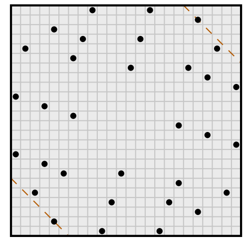
$m = 21$, $\beta = 37$



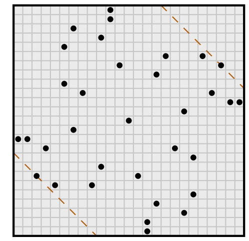
$m = 22,\ \beta = 29$



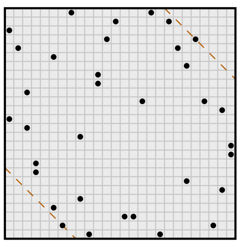
$m = 23,\ \beta = 33$



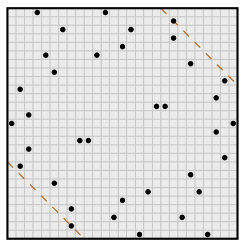
$m = 24,\ \beta = 37$



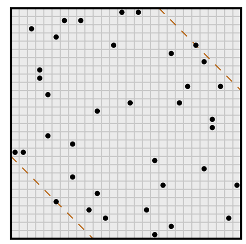
$m = 25,\ \beta = 33$



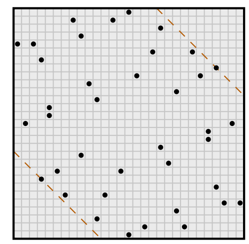
$m = 26,\ \beta = 37$



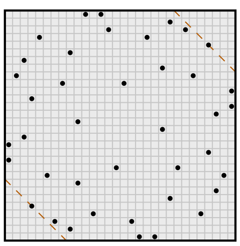
$m = 27,\ \beta = 37$



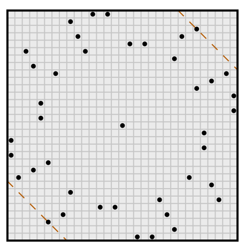
$m = 28,\ \beta = 37$



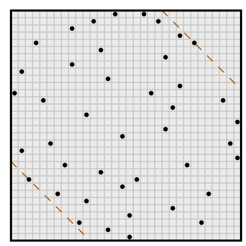
$m = 29,\ \beta = 37$



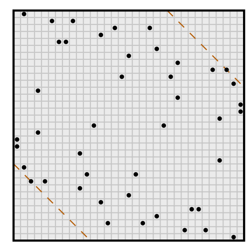
$m = 30,\ \beta = 45$



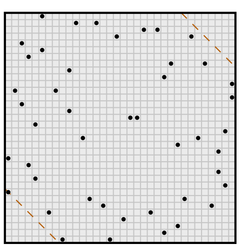
$m = 31,\ \beta = 47$



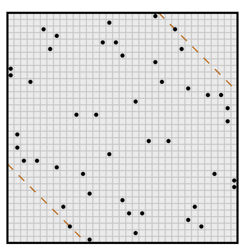
$m = 32,\ \beta = 43$



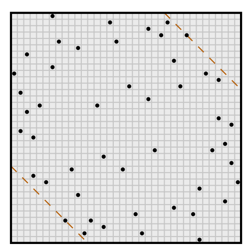
$m = 33,\ \beta = 45$



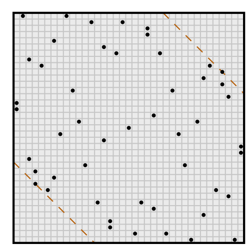
$m = 34,\ \beta = 53$



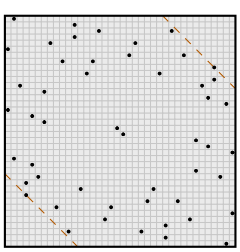
$m = 35,\ \beta = 47$



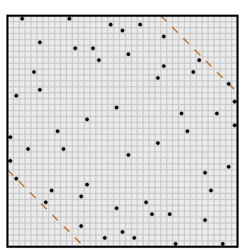
$m = 36,\ \beta = 49$



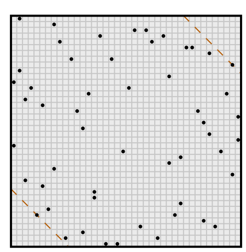
$m = 37,\ \beta = 49$



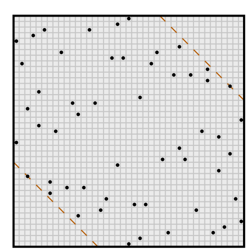
$m = 38,\ \beta = 53$



$m = 39,\ \beta = 53$



$m = 40,\ \beta = 61$



$m = 41,\ \beta = 53$

Coordinates, with $(0,0)$ at the top-left as in Figure 2.

$m = 6 : \{\ (0,3),\ (0,4),\ (1,0),\ (2,0),\ (3,5),\ (4,5),\ (5,1),\ (5,2)\ \}.$

$m = 7 : \{\ (0,2),\ (0,5),\ (1,0),\ (2,6),\ (3,3),\ (4,0),\ (5,6),\ (6,1),\ (6,4)\ \}.$

$m = 8 : \{\ (0,0),\ (1,2),\ (1,3),\ (2,6),\ (3,6),\ (4,1),\ (5,1),\ (6,4),\ (6,5),\ (7,7)\ \}.$

$m = 9 : \{\ (0,3),\ (0,4),\ (1,0),\ (2,0),\ (3,5),\ (3,8),\ (4,1),\ (5,1),\ (6,6),\ (6,7),\ (7,2),\ (8,2)\ \}.$

$m = 10 : \{\ (0,1),\ (0,5),\ (1,7),\ (2,2),\ (3,0),\ (4,3),\ (4,6),\ (5,8),\ (5,9),\ (6,0),\ (7,2),\ (8,4),\ (9,7)\ \}.$

$m = 11 : \{\ (0,1),\ (0,6),\ (1,4),\ (2,0),\ (2,2),\ (3,8),\ (4,9),\ (5,4),\ (6,10),\ (7,3),\ (8,10),\ (9,3),\ (10,5),\ (10,7)\ \}.$

$m = 12 : \{\ (0,4),\ (0,6),\ (1,2),\ (1,8),\ (2,0),\ (3,11),\ (4,1),\ (5,10),\ (6,1),\ (7,10),\ (8,0),\ (9,11),\ (10,3),\ (10,9),\ (11,5),\ (11,7)\ \}.$

$m = 13 : \{\ (0,7),\ (0,9),\ (1,2),\ (1,4),\ (2,0),\ (3,11),\ (4,0),\ (5,11),\ (6,6),\ (7,1),\ (8,12),\ (9,1),\ (10,12),\ (11,8),\ (11,10),\ (12,3),\ (12,5)\ \}.$

$m = 14 : \{\ (0,2),\ (1,6),\ (2,1),\ (3,10),\ (3,11),\ (4,5),\ (4,8),\ (5,3),\ (6,0),\ (6,12),\ (7,3),\ (8,7),\ (9,2),\ (10,13),\ (11,6),\ (12,4),\ (12,9),\ (13,13)\ \}.$

$m = 15 : \{\ (0,7),\ (1,9),\ (1,10),\ (2,1),\ (3,13),\ (4,0),\ (5,3),\ (5,8),\ (6,0),\ (7,2),\ (7,12),\ (8,14),\ (9,6),\ (9,11),\ (10,14),\ (11,1),\ (12,13),\ (13,4),\ (13,5),\ (14,7)\ \}.$

$m = 16 : \{\ (0,8),\ (0,12),\ (1,6),\ (2,4),\ (2,9),\ (3,1),\ (3,13),\ (4,10),\ (5,0),\ (6,15),\ (7,3),\ (8,11),\ (9,0),\ (10,3),\ (11,15),\ (12,6),\ (13,2),\ (13,14),\ (14,11),\ (15,5),\ (15,7)\ \}.$

$m = 17 : \{\ (0,7),\ (0,10),\ (1,12),\ (1,14),\ (2,0),\ (3,15),\ (4,8),\ (5,0),\ (6,3),\ (6,5),\ (7,1),\ (9,15),\ (10,11),\ (10,13),\ (11,16),\ (12,8),\ (13,1),\ (14,16),\ (15,2),\ (15,4),\ (16,6),\ (16,9)\ \}.$

$m = 18 : \{\ (0,9),\ (0,11),\ (1,4),\ (1,15),\ (2,1),\ (3,10),\ (4,16),\ (5,10),\ (6,0),\ (7,3),\ (7,5),\ (8,0),\ (9,17),\ (10,12),\ (10,14),\ (11,17),\ (12,7),\ (13,1),\ (14,7),\ (15,16),\ (16,2),\ (16,13),\ (17,6),\ (17,8)\ \}.$

$m = 19 : \{\ (0,14),\ (1,7),\ (1,10),\ (2,5),\ (2,15),\ (3,2),\ (4,0),\ (4,12),\ (5,16),\ (6,4),\ (7,17),\ (8,1),\ (9,9),\ (10,17),\ (11,1),\ (12,14),\ (13,2),\ (14,6),\ (14,18),\ (15,16),\ (16,3),\ (16,13),\ (17,8),\ (17,11),\ (18,4)\ \}.$

$m = 20 : \{\ (0,13),\ (0,14),\ (1,2),\ (1,4),\ (2,8),\ (2,9),\ (3,18),\ (4,16),\ (5,1),\ (6,16),\ (7,12),\ (8,0),\ (9,0),\ (10,19),\ (11,19),\ (12,7),\ (13,3),\ (14,18),\ (15,3),\ (16,1),\ (17,10),\ (17,11),\ (18,15),\ (18,17),\ (19,5),\ (19,6)\ \}.$

$m = 21 : \{\ (0,6),\ (0,18),\ (1,8),\ (1,11),\ (2,3),\ (3,5),\ (3,7),\ (4,17),\ (5,20),\ (6,20),\ (7,10),\ (8,16),\ (9,4),\ (10,1),\ (10,19),\ (11,16),\ (12,4),\ (13,10),\ (14,0),\ (15,0),\ (16,3),\ (17,13),\ (17,15),\ (18,17),\ (19,9),\ (19,12),\ (20,2),\ (20,14)\ \}.$

$m = 22 : \{\ (0,0),\ (1,6),\ (2,11),\ (2,13),\ (3,16),\ (4,18),\ (5,7),\ (5,15),\ (6,18),\ (7,3),\ (7,10),\ (8,0),\ (9,16),\ (10,1),\ (11,5),\ (11,17),\ (12,20),\ (13,1),\ (14,9),\ (15,12),\ (16,2),\ (16,14),\ (17,4),\ (17,21),\ (18,8),\ (18,19),\ (19,12),\ (20,9),\ (21,20)\ \}.$

$m = 23 : \{\ (0,9),\ (0,14),\ (1,5),\ (2,5),\ (3,15),\ (4,19),\ (5,6),\ (5,10),\ (6,22),\ (7,18),\ (7,20),\ (8,0),\ (9,15),\ (10,3),\ (11,1),\ (11,21),\ (12,19),\ (13,7),\ (14,22),\ (15,2),\ (15,4),\ (16,0),\ (17,12),\ (17,16),\ (18,3),\ (19,7),\ (20,17),\ (21,17),\ (22,8),\ (22,13)\ \}.$

$m = 24 : \{\ (0,8),\ (0,14),\ (1,19),\ (2,4),\ (3,7),\ (3,13),\ (4,1),\ (4,21),\ (5,6),\ (6,12),\ (6,18),\ (7,20),\ (8,23),\ (9,0),\ (10,3),\ (11,6),\ (12,17),\ (13,20),\ (14,23),\ (15,0),\ (16,3),\ (17,5),\ (17,11),\ (18,17),\ (19,2),\ (19,22),\ (20,10),\ (20,16),\ (21,19),\ (22,4),\ (23,9),\ (23,15)\ \}.$

$m = 25 : \{\ (0,10),\ (1,10),\ (2,6),\ (3,9),\ (4,5),\ (5,16),\ (5,20),\ (6,11),\ (6,22),\ (7,15),\ (8,5),\ (9,7),\ (9,21),\ (10,23),\ (10,24),\ (11,18),\ (12,12),\ (13,6),\ (14,0),\ (14,1),\ (15,3),\ (15,17),\ (16,19),\ (17,9),\ (18,2),\ (18,13),\ (19,4),\ (19,8),\ (20,19),\ (21,15),\ (22,18),\ (23,14),\ (24,14)\ \}.$

$m = 26 : \{\ (0,7),\ (0,16),\ (1,12),\ (1,18),\ (2,0),\ (3,11),\ (3,21),\ (4,1),\ (4,19),\ (5,5),\ (6,20),\ (7,10),\ (8,10),\ (9,2),\ (10,15),\ (10,22),\ (11,24),\ (12,0),\ (13,2),\ (14,8),\ (15,25),\ (16,25),\ (17,3),\ (18,3),\ (19,20),\ (20,24),\ (21,8),\ (22,5),\ (23,13),\ (23,14),\ (24,6),\ (24,23),\ (25,9),\ (25,17)\ \}.$

$m = 27 : \{\ (0,3),\ (0,11),\ (1,19),\ (2,6),\ (2,14),\ (3,19),\ (4,13),\ (5,4),\ (5,10),\ (6,21),\ (7,5),\ (8,25),\ (9,1),\ (10,24),\ (11,17),\ (11,18),\ (12,2),\ (13,0),\ (13,26),\ (14,24),\ (15,8),\ (15,9),\ (16,2),\ (17,25),\ (18,1),\ (19,21),\ (20,5),\ (21,16),\ (21,22),\ (22,13),\ (23,7),\ (24,12),\ (24,20),\ (25,7),\ (26,15),\ (26,23)\ \}.$

$m = 28 : \{\ (0,13),\ (0,15),\ (1,6),\ (1,8),\ (2,2),\ (3,5),\ (4,12),\ (4,22),\ (5,19),\ (6,23),\ (7,3),\ (8,3),\ (9,21),\ (9,25),\ (10,4),\ (11,14),\ (11,20),\ (12,10),\ (13,24),\ (14,24),\ (15,4),\ (16,7),\ (17,0),\ (17,1),\ (18,17),\ (19,23),\ (20,7),\ (21,18),\ (21,27),\ (22,10),\ (23,5),\ (24,9),\ (24,16),\ (25,11),\ (25,26),\ (26,19),\ (27,17)\ \}.$

$m = 29 : \{\ (0,14),\ (1,7),\ (1,12),\ (2,18),\ (3,8),\ (4,0),\ (4,2),\ (5,17),\ (5,22),\ (6,3),\ (7,25),\ (8,15),\ (8,23),\ (9,9),$

$(10,20)$, $(11,10)$, $(12,4)$, $(13,4)$, $(14,1)$, $(14,27)$, $(15,24)$, $(16,24)$, $(17,18)$, $(18,8)$, $(19,19)$, $(20,5)$, $(20,13)$, $(21,3)$, $(22,25)$, $(23,6)$, $(23,11)$, $(24,26)$, $(24,28)$, $(25,20)$, $(26,10)$, $(27,16)$, $(27,21)$, $(28,14)$ $\}$.

$m = 30 : \{$ $(0,10)$, $(0,12)$, $(1,21)$, $(2,13)$, $(2,23)$, $(3,4)$, $(3,18)$, $(4,26)$, $(5,8)$, $(6,2)$, $(7,20)$, $(8,1)$, $(8,24)$, $(9,7)$, $(9,15)$, $(10,29)$, $(11,3)$, $(12,29)$, $(13,27)$, $(14,9)$, $(15,20)$, $(16,2)$, $(17,0)$, $(18,26)$, $(19,0)$, $(20,14)$, $(20,22)$, $(21,5)$, $(21,28)$, $(22,9)$, $(23,27)$, $(24,21)$, $(25,3)$, $(26,11)$, $(26,25)$, $(27,6)$, $(27,16)$, $(28,8)$, $(29,17)$, $(29,19)$ $\}$.

$m = 31 : \{$ $(0,11)$, $(0,13)$, $(1,8)$, $(2,25)$, $(3,9)$, $(3,23)$, $(4,16)$, $(4,18)$, $(5,2)$, $(5,10)$, $(6,22)$, $(7,3)$, $(8,6)$, $(8,29)$, $(9,27)$, $(10,25)$, $(11,30)$, $(12,4)$, $(13,30)$, $(14,4)$, $(15,15)$, $(16,26)$, $(17,0)$, $(18,26)$, $(19,0)$, $(20,5)$, $(21,3)$, $(22,1)$, $(22,24)$, $(23,27)$, $(24,8)$, $(25,20)$, $(25,28)$, $(26,12)$, $(26,14)$, $(27,7)$, $(27,21)$, $(28,5)$, $(29,22)$, $(30,17)$, $(30,19)$ $\}$.

$m = 32 : \{$ $(0,14)$, $(0,18)$, $(1,11)$, $(1,20)$, $(2,8)$, $(3,23)$, $(4,3)$, $(4,25)$, $(5,12)$, $(6,21)$, $(7,8)$, $(8,1)$, $(9,13)$, $(10,23)$, $(11,0)$, $(11,19)$, $(12,4)$, $(12,29)$, $(13,22)$, $(14,10)$, $(15,31)$, $(16,21)$, $(17,15)$, $(18,5)$, $(18,30)$, $(19,1)$, $(20,31)$, $(21,7)$, $(21,24)$, $(22,12)$, $(23,2)$, $(23,17)$, $(24,15)$, $(25,6)$, $(25,27)$, $(26,10)$, $(27,22)$, $(28,16)$, $(29,9)$, $(29,26)$, $(30,13)$, $(31,16)$ $\}$.

$m = 33 : \{$ $(0,1)$, $(1,5)$, $(1,8)$, $(2,14)$, $(2,19)$, $(3,12)$, $(4,6)$, $(4,7)$, $(5,20)$, $(6,16)$, $(7,23)$, $(8,28)$, $(8,30)$, $(9,15)$, $(9,22)$, $(10,31)$, $(11,3)$, $(12,23)$, $(13,32)$, $(14,32)$, $(15,29)$, $(16,11)$, $(16,21)$, $(17,3)$, $(18,0)$, $(19,0)$, $(20,9)$, $(21,29)$, $(22,1)$, $(23,10)$, $(23,17)$, $(24,2)$, $(24,4)$, $(25,9)$, $(26,16)$, $(27,12)$, $(28,25)$, $(28,26)$, $(29,20)$, $(30,13)$, $(30,18)$, $(31,24)$, $(31,27)$, $(32,31)$ $\}$.

$m = 34 : \{$ $(0,5)$, $(1,10)$, $(1,13)$, $(2,20)$, $(2,22)$, $(3,16)$, $(3,27)$, $(4,2)$, $(5,5)$, $(6,3)$, $(7,24)$, $(7,29)$, $(8,9)$, $(9,23)$, $(10,33)$, $(11,1)$, $(11,7)$, $(12,33)$, $(13,2)$, $(14,9)$, $(15,18)$, $(15,19)$, $(16,4)$, $(17,32)$, $(18,11)$, $(18,28)$, $(19,25)$, $(20,31)$, $(21,0)$, $(22,3)$, $(23,31)$, $(24,4)$, $(25,32)$, $(26,0)$, $(27,12)$, $(27,26)$, $(28,14)$, $(28,30)$, $(29,6)$, $(29,21)$, $(30,17)$, $(31,25)$, $(32,23)$, $(33,8)$, $(33,15)$ $\}$.

$m = 35 : \{$ $(0,22)$, $(1,15)$, $(2,5)$, $(2,25)$, $(3,7)$, $(4,14)$, $(4,16)$, $(5,6)$, $(5,26)$, $(6,17)$, $(7,22)$, $(8,0)$, $(9,0)$, $(10,3)$, $(10,23)$, $(11,27)$, $(12,30)$, $(12,32)$, $(13,19)$, $(14,33)$, $(15,10)$, $(15,13)$, $(16,33)$, $(18,1)$, $(19,21)$, $(19,24)$, $(20,1)$, $(21,15)$, $(22,2)$, $(22,4)$, $(23,7)$, $(24,11)$, $(24,31)$, $(25,34)$, $(26,34)$, $(27,12)$, $(28,17)$, $(29,8)$, $(29,28)$, $(30,18)$, $(30,20)$, $(31,27)$, $(32,9)$, $(32,29)$, $(33,19)$, $(34,12)$ $\}$.

$m = 36 : \{$ $(0,6)$, $(1,15)$, $(1,24)$, $(2,21)$, $(3,23)$, $(3,27)$, $(4,7)$, $(4,16)$, $(5,10)$, $(6,2)$, $(7,25)$, $(8,6)$, $(9,0)$, $(9,30)$, $(10,32)$, $(11,18)$, $(11,26)$, $(12,1)$, $(13,21)$, $(14,4)$, $(14,13)$, $(15,2)$, $(16,32)$, $(17,34)$, $(18,1)$, $(19,3)$, $(20,33)$, $(21,22)$, $(21,31)$, $(22,14)$, $(23,34)$, $(24,9)$, $(24,17)$, $(25,3)$, $(26,5)$, $(26,35)$, $(27,29)$, $(28,10)$, $(29,33)$, $(30,25)$, $(31,19)$, $(31,28)$, $(32,8)$, $(32,12)$, $(33,14)$, $(34,11)$, $(34,20)$, $(35,29)$ $\}$.

$m = 37 : \{$ $(0,1)$, $(0,8)$, $(1,12)$, $(1,17)$, $(2,21)$, $(3,21)$, $(4,6)$, $(5,14)$, $(6,16)$, $(6,23)$, $(7,2)$, $(8,4)$, $(8,31)$, $(9,33)$, $(10,30)$, $(11,33)$, $(12,9)$, $(12,25)$, $(13,34)$, $(14,0)$, $(15,0)$, $(16,22)$, $(17,10)$, $(17,29)$, $(18,18)$, $(19,7)$, $(19,26)$, $(20,14)$, $(21,36)$, $(22,36)$, $(23,2)$, $(24,11)$, $(24,27)$, $(25,3)$, $(26,6)$, $(27,3)$, $(28,5)$, $(28,32)$, $(29,34)$, $(30,13)$, $(30,20)$, $(31,22)$, $(32,30)$, $(33,15)$, $(34,15)$, $(35,19)$, $(35,24)$, $(36,28)$, $(36,35)$ $\}$.

$m = 38 : \{$ $(0,1)$, $(1,11)$, $(2,15)$, $(2,27)$, $(3,11)$, $(4,7)$, $(4,21)$, $(5,0)$, $(6,20)$, $(6,29)$, $(7,9)$, $(7,14)$, $(8,34)$, $(9,13)$, $(9,25)$, $(10,34)$, $(11,2)$, $(11,32)$, $(12,6)$, $(13,33)$, $(14,36)$, $(15,0)$, $(16,4)$, $(17,6)$, $(18,18)$, $(19,19)$, $(20,31)$, $(21,33)$, $(22,37)$, $(23,1)$, $(24,4)$, $(25,31)$, $(26,5)$, $(26,35)$, $(27,3)$, $(28,12)$, $(28,24)$, $(29,3)$, $(30,23)$, $(30,28)$, $(31,8)$, $(31,17)$, $(32,37)$, $(33,16)$, $(33,30)$, $(34,26)$, $(35,10)$, $(35,22)$, $(36,26)$, $(37,36)$ $\}$.

$m = 39 : \{$ $(0,2)$, $(0,10)$, $(1,17)$, $(1,22)$, $(2,19)$, $(3,26)$, $(4,5)$, $(5,11)$, $(5,14)$, $(6,20)$, $(7,15)$, $(7,32)$, $(8,26)$, $(9,4)$, $(9,31)$, $(10,25)$, $(11,37)$, $(12,5)$, $(13,1)$, $(14,38)$, $(15,18)$, $(16,29)$, $(16,35)$, $(17,13)$, $(18,38)$, $(19,8)$, $(19,30)$, $(20,0)$, $(21,25)$, $(22,3)$, $(22,9)$, $(23,20)$, $(24,0)$, $(25,37)$, $(26,33)$, $(27,1)$, $(28,13)$, $(29,7)$, $(29,34)$, $(30,12)$, $(31,6)$, $(31,23)$, $(32,18)$, $(33,24)$, $(33,27)$, $(34,33)$, $(35,12)$, $(36,19)$, $(37,16)$, $(37,21)$, $(38,28)$, $(38,36)$ $\}$.

$m = 40 : \{$ $(0,1)$, $(1,7)$, $(2,21)$, $(2,23)$, $(3,15)$, $(3,26)$, $(4,8)$, $(4,24)$, $(5,30)$, $(5,31)$, $(6,34)$, $(7,10)$, $(7,17)$, $(8,38)$, $(9,1)$, $(10,27)$, $(11,0)$, $(12,3)$, $(12,20)$, $(13,13)$, $(13,37)$, $(14,2)$, $(15,5)$, $(16,11)$, $(16,32)$, $(17,39)$, $(18,33)$, $(19,12)$, $(20,34)$, $(21,39)$, $(22,0)$, $(23,19)$, $(23,36)$, $(24,29)$, $(25,27)$, $(26,7)$, $(27,38)$, $(28,2)$, $(29,5)$, $(30,14)$, $(31,14)$, $(32,29)$, $(33,6)$, $(34,4)$, $(34,28)$, $(35,33)$, $(36,22)$, $(36,35)$, $(37,12)$, $(38,9)$, $(38,25)$, $(39,16)$, $(39,18)$ $\}$.

$m = 41 : \{$ $(0,20)$, $(1,18)$, $(2,5)$, $(2,13)$, $(3,3)$, $(4,0)$, $(5,29)$, $(6,8)$, $(6,25)$, $(7,17)$, $(7,19)$, $(8,1)$, $(8,24)$, $(9,34)$, $(10,28)$, $(10,31)$, $(11,34)$, $(12,38)$, $(13,4)$, $(14,22)$, $(15,10)$, $(15,14)$, $(16,2)$, $(17,11)$, $(18,40)$, $(19,4)$, $(20,7)$, $(20,33)$, $(21,36)$, $(22,0)$, $(23,29)$, $(24,38)$, $(25,26)$, $(25,30)$, $(26,18)$, $(27,36)$, $(28,2)$, $(29,6)$, $(30,9)$, $(30,12)$, $(31,6)$, $(32,16)$, $(32,39)$, $(33,21)$, $(33,23)$, $(34,15)$, $(34,32)$, $(35,11)$, $(36,40)$, $(37,37)$, $(38,27)$, $(38,35)$, $(39,22)$, $(40,20)$ $\}$.

B Constructions for $n \leq 215$

Table 4 records one q_1-optimal construction for each $6 \leq n \leq 215$, in the notation of Section 2. Here T_s is the placement (4) of s non-attacking queens on the torus; C_n is a channelled bound-optimal board of size n, as in Tables 2 and 3 and Appendix C; an asterisk is an explicit stored board from Appendix A (the corners of Table 1 when $n \leq 29$). The sum $X + m$ is the corner paste of Lemma 5, and $X + 1$ is the isolate gadget. From $n = 216$ the two product families of Section 4 take over.

Table 4: One q_1-construction for each $6 \leq n \leq 215$.

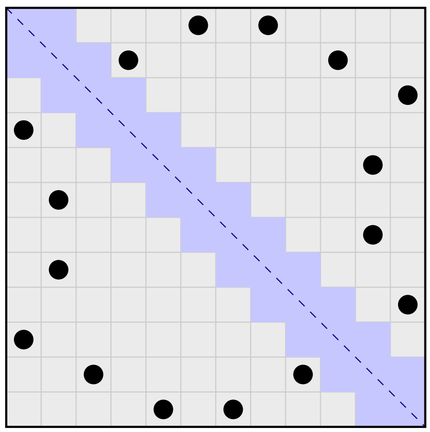
n : construction	n : construction	n : construction
$6 : S_6$	$7 : *$	$8 : *$
$9 : *$	$10 : *$	$11 : *$
$12 : C_{12}$	$13 : *$	$14 : *$
$15 : C_{15}$	$16 : *$	$17 : *$
$18 : C_{18}$	$19 : *$	$20 : *$
$21 : *$	$22 : *$	$23 : *$
$24 : C_{24}$	$25 : *$	$26 : *$
$27 : *$	$28 : *$	$29 : *$
$30 : S_6 \otimes T_5$	$31 : *$	$32 : *$
$33 : *$	$34 : *$	$35 : *$
$36 : *$	$37 : *$	$38 : *$
$39 : *$	$40 : *$	$41 : *$
$42 : S_6 \otimes T_7$	$43 : S_6 \otimes T_7 + 1$	$44 : S_6 \otimes T_7 + 2$
$45 : C_{45}$	$46 : C_{45} + 1$	$47 : C_{45} + 2$
$48 : C_{48}$	$49 : C_{48} + 1$	$50 : C_{48} + 2$
$51 : C_{45} + 6$	$52 : C_{45} + 6 + 1$	$53 : C_{45} + 8$
$54 : C_{48} + 6$	$55 : C_{48} + 7$	$56 : C_{48} + 8$
$57 : C_{48} + 9$	$58 : C_{48} + 10$	$59 : C_{48} + 11$
$60 : C_{12} \otimes T_5$	$61 : C_{12} \otimes T_5 + 1$	$62 : C_{12} \otimes T_5 + 2$
$63 : C_{63}$	$64 : C_{63} + 1$	$65 : C_{63} + 2$
$66 : S_6 \otimes T_{11}$	$67 : C_{12} \otimes T_5 + 7$	$68 : S_6 \otimes T_{11} + 2$
$69 : C_{12} \otimes T_5 + 9$	$70 : C_{12} \otimes T_5 + 10$	$71 : C_{12} \otimes T_5 + 11$
$72 : S_6 \otimes T_{11} + 6$	$73 : S_6 \otimes T_{11} + 7$	$74 : S_6 \otimes T_{11} + 8$
$75 : C_{15} \otimes T_5$	$76 : C_{15} \otimes T_5 + 1$	$77 : C_{15} \otimes T_5 + 2$
$78 : S_6 \otimes T_{13}$	$79 : S_6 \otimes T_{13} + 1$	$80 : S_6 \otimes T_{13} + 2$
$81 : C_{15} \otimes T_5 + 6$	$82 : C_{15} \otimes T_5 + 7$	$83 : C_{15} \otimes T_5 + 8$
$84 : C_{12} \otimes T_7$	$85 : S_6 \otimes T_{13} + 7$	$86 : S_6 \otimes T_{13} + 8$
$87 : S_6 \otimes T_{13} + 9$	$88 : S_6 \otimes T_{13} + 10$	$89 : S_6 \otimes T_{13} + 11$
$90 : C_{18} \otimes T_5$	$91 : C_{12} \otimes T_7 + 7$	$92 : C_{12} \otimes T_7 + 8$
$93 : C_{12} \otimes T_7 + 9$	$94 : C_{12} \otimes T_7 + 10$	$95 : C_{12} \otimes T_7 + 11$
$96 : C_{12} \otimes T_7 + 12$	$97 : C_{12} \otimes T_7 + 13$	$98 : C_{12} \otimes T_7 + 14$
$99 : C_{12} \otimes T_7 + 15$	$100 : C_{18} \otimes T_5 + 10$	$101 : C_{18} \otimes T_5 + 11$
$102 : S_6 \otimes T_{17}$	$103 : S_6 \otimes T_{17} + 1$	$104 : S_6 \otimes T_{17} + 2$
$105 : C_{15} \otimes T_7$	$106 : C_{15} \otimes T_7 + 1$	$107 : C_{15} \otimes T_7 + 2$
$108 : S_6 \otimes T_{17} + 6$	$109 : S_6 \otimes T_{17} + 7$	$110 : S_6 \otimes T_{17} + 8$
$111 : S_6 \otimes T_{17} + 9$	$112 : S_6 \otimes T_{17} + 10$	$113 : S_6 \otimes T_{17} + 11$
$114 : S_6 \otimes T_{19}$	$115 : C_{15} \otimes T_7 + 10$	$116 : S_6 \otimes T_{17} + 14$
$117 : C_{15} \otimes T_7 + 12$	$118 : C_{15} \otimes T_7 + 13$	$119 : C_{15} \otimes T_7 + 14$
$120 : C_{24} \otimes T_5$	$121 : S_6 \otimes T_{19} + 7$	$122 : S_6 \otimes T_{19} + 8$

Continued.

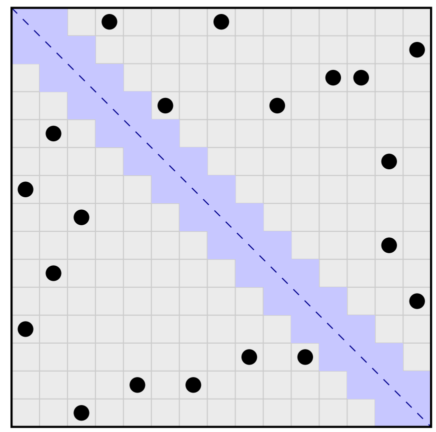
n : construction	n : construction	n : construction
$123: S_6 \otimes T_{19} + 9$	$124: S_6 \otimes T_{19} + 10$	$125: S_6 \otimes T_{19} + 11$
$126: C_{18} \otimes T_7$	$127: S_6 \otimes T_{19} + 13$	$128: S_6 \otimes T_{19} + 14$
$129: C_{24} \otimes T_5 + 9$	$130: C_{24} \otimes T_5 + 10$	$131: C_{24} \otimes T_5 + 11$
$132: C_{12} \otimes T_{11}$	$133: C_{24} \otimes T_5 + 13$	$134: C_{24} \otimes T_5 + 14$
$135: C_{24} \otimes T_5 + 15$	$136: C_{24} \otimes T_5 + 16$	$137: C_{18} \otimes T_7 + 11$
$138: S_6 \otimes T_{23}$	$139: C_{18} \otimes T_7 + 13$	$140: S_6 \otimes T_{23} + 2$
$141: C_{18} \otimes T_7 + 15$	$142: C_{12} \otimes T_{11} + 10$	$143: C_{12} \otimes T_{11} + 11$
$144: S_6 \otimes T_{23} + 6$	$145: S_6 \otimes T_{23} + 7$	$146: S_6 \otimes T_{23} + 8$
$147: S_6 \otimes T_{23} + 9$	$148: S_6 \otimes T_{23} + 10$	$149: S_6 \otimes T_{23} + 11$
$150: S_6 \otimes T_{25}$	$151: S_6 \otimes T_{23} + 13$	$152: S_6 \otimes T_{23} + 14$
$153: S_6 \otimes T_{23} + 15$	$154: C_{12} \otimes T_{11} + 22$	$155: C_{12} \otimes T_{11} + 23$
$156: C_{12} \otimes T_{13}$	$157: S_6 \otimes T_{25} + 7$	$158: S_6 \otimes T_{25} + 8$
$159: S_6 \otimes T_{25} + 9$	$160: S_6 \otimes T_{25} + 10$	$161: S_6 \otimes T_{25} + 11$
$162: S_6 \otimes T_{25} + 12$	$163: S_6 \otimes T_{25} + 13$	$164: S_6 \otimes T_{25} + 14$
$165: C_{15} \otimes T_{11}$	$166: S_6 \otimes T_{25} + 16$	$167: C_{12} \otimes T_{13} + 11$
$168: C_{24} \otimes T_7$	$169: C_{12} \otimes T_{13} + 13$	$170: C_{12} \otimes T_{13} + 14$
$171: C_{12} \otimes T_{13} + 15$	$172: C_{12} \otimes T_{13} + 16$	$173: C_{12} \otimes T_{13} + 17$
$174: S_6 \otimes T_{29}$	$175: C_{12} \otimes T_{13} + 19$	$176: S_6 \otimes T_{29} + 2$
$177: C_{12} \otimes T_{13} + 21$	$178: C_{12} \otimes T_{13} + 22$	$179: C_{12} \otimes T_{13} + 23$
$180: S_6 \otimes T_{29} + 6$	$181: S_6 \otimes T_{29} + 7$	$182: S_6 \otimes T_{29} + 8$
$183: S_6 \otimes T_{29} + 9$	$184: S_6 \otimes T_{29} + 10$	$185: S_6 \otimes T_{29} + 11$
$186: S_6 \otimes T_{31}$	$187: S_6 \otimes T_{29} + 13$	$188: S_6 \otimes T_{29} + 14$
$189: S_6 \otimes T_{29} + 15$	$190: S_6 \otimes T_{29} + 16$	$191: S_6 \otimes T_{29} + 17$
$192: S_6 \otimes T_{29} + 18$	$193: S_6 \otimes T_{29} + 19$	$194: S_6 \otimes T_{31} + 8$
$195: C_{15} \otimes T_{13}$	$196: S_6 \otimes T_{29} + 22$	$197: S_6 \otimes T_{31} + 11$
$198: C_{18} \otimes T_{11}$	$199: S_6 \otimes T_{31} + 13$	$200: S_6 \otimes T_{31} + 14$
$201: S_6 \otimes T_{31} + 15$	$202: S_6 \otimes T_{31} + 16$	$203: S_6 \otimes T_{31} + 17$
$204: C_{12} \otimes T_{17}$	$205: S_6 \otimes T_{31} + 19$	$206: S_6 \otimes T_{31} + 20$
$207: C_{15} \otimes T_{13} + 12$	$208: S_6 \otimes T_{31} + 22$	$209: C_{15} \otimes T_{13} + 14$
$210: S_6 \otimes T_{35}$	$211: C_{15} \otimes T_{13} + 16$	$212: S_6 \otimes T_{35} + 2$
$213: C_{15} \otimes T_{13} + 18$	$214: C_{15} \otimes T_{13} + 19$	$215: C_{15} \otimes T_{13} + 20$

C The channelled seeds

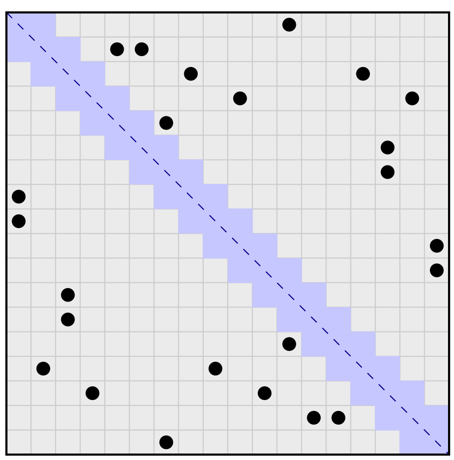
The boards below are the channelled placements C_n named in Tables 2 and 3. Shaded cells are the empty channel $|r - c| < (\chi + 1)/2$; the dashed line is the main diagonal $r = c$. These are not the narrow corners of Appendix A: the search here maximises χ rather than minimising β. The four tensor factors of size at most 24 come first, then the leftover bases with $n > 41$. Coordinates with $(0, 0)$ at the top-left follow the drawings.



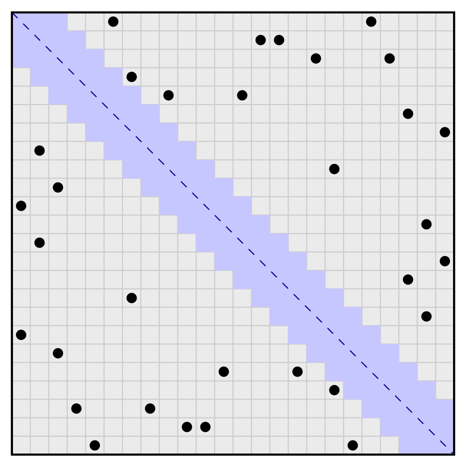
C_{12}, $\chi = 3$, $|C_{12}| = 16$



C_{15}, $\chi = 3$, $|C_{15}| = 20$

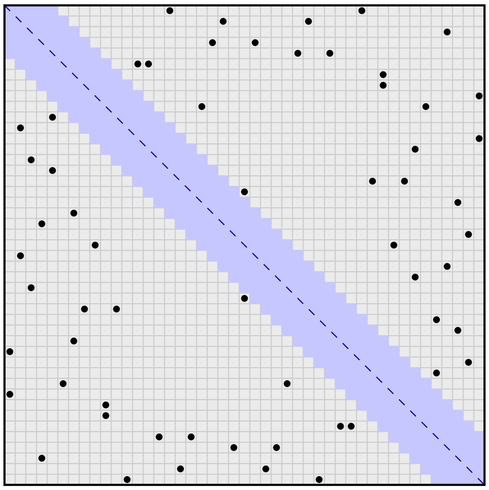


C_{18}, $\chi = 3$, $|C_{18}| = 24$

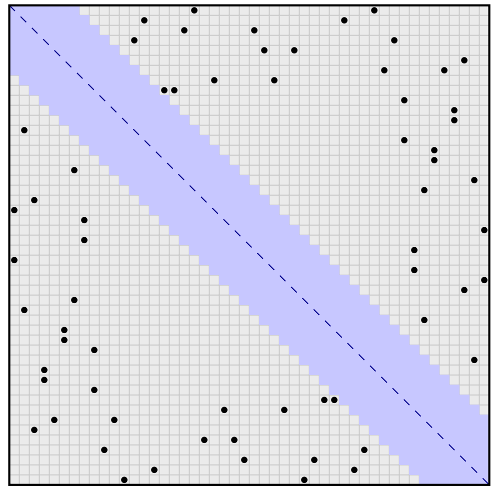


C_{24}, $\chi = 5$, $|C_{24}| = 32$

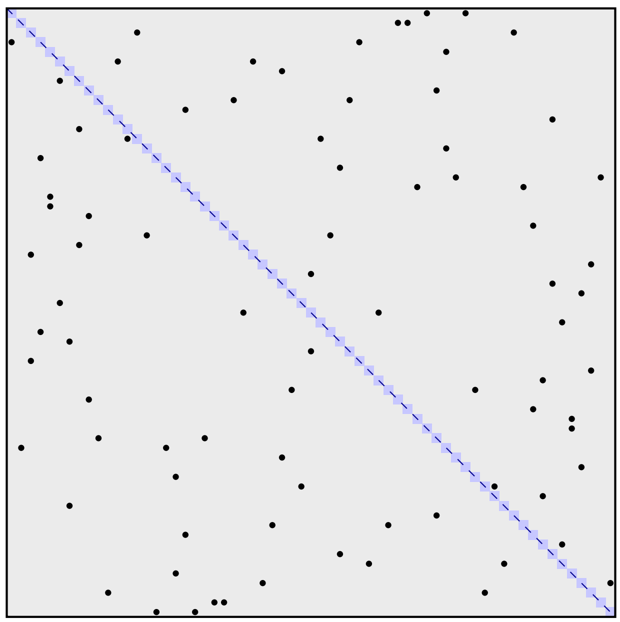
The leftover bases with $n > 41$. None of these is tensored.



C_{45}, $\chi = 9$, $|C_{45}| = 60$



C_{48}, $\chi = 13$, $|C_{48}| = 64$



C_{63}, $\chi = 1$, $|C_{63}| = 84$

Coordinates, with $(0,0)$ at the top-left as in Figure 2. The four small tensor factors are listed first.

$C_{12} = \{\ (0,5), (0,7), (1,3), (1,9), (2,11), (3,0), (4,10), (5,1), (6,10), (7,1), (8,11), (9,0), (10,2), (10,8), (11,4), (11,6)\ \}$.

$C_{15} = \{\ (0,3), (0,7), (1,14), (2,11), (2,12), (3,5), (3,9), (4,1), (5,13), (6,0), (7,2), (8,13), (9,1), (10,14), (11,0), (12,8), (12,10), (13,4), (13,6), (14,2)\ \}$.

$C_{18} = \{\ (0,11), (1,4), (1,5), (2,7), (2,14), (3,9), (3,16), (4,6), (5,15), (6,15), (7,0), (8,0), (9,17), (10,17), (11,2), (12,2), (13,11), (14,1), (14,8), (15,3), (15,10), (16,12), (16,13), (17,6)\ \}$.

$C_{24} = \{\ (0,5), (0,19), (1,13), (1,14), (2,16), (2,20), (3,6), (4,8), (4,12), (5,21), (6,23), (7,1), (8,17), (9,2), (10,0), (11,22), (12,1), (13,23), (14,21), (15,6), (16,22), (17,0), (18,2), (19,11), (19,15), (20,17), (21,3), (21,7), (22,9), (22,10), (23,4), (23,18)\ \}$.

$C_{45} = \{$ $(0,15)$, $(0,33)$, $(1,20)$, $(1,28)$, $(2,41)$, $(3,19)$, $(3,23)$, $(4,27)$, $(4,30)$, $(5,12)$, $(5,13)$, $(6,35)$, $(7,35)$, $(8,44)$, $(9,18)$, $(9,39)$, $(10,4)$, $(11,1)$, $(12,44)$, $(13,38)$, $(14,2)$, $(15,4)$, $(16,34)$, $(16,37)$, $(17,22)$, $(18,42)$, $(19,6)$, $(20,3)$, $(21,43)$, $(22,8)$, $(22,36)$, $(23,1)$, $(24,41)$, $(25,38)$, $(26,2)$, $(27,22)$, $(28,7)$, $(28,10)$, $(29,40)$, $(30,42)$, $(31,6)$, $(32,0)$, $(33,43)$, $(34,40)$, $(35,5)$, $(35,26)$, $(36,0)$, $(37,9)$, $(38,9)$, $(39,31)$, $(39,32)$, $(40,14)$, $(40,17)$, $(41,21)$, $(41,25)$, $(42,3)$, $(43,16)$, $(43,24)$, $(44,11)$, $(44,29)$ $\}$.

$C_{48} = \{$ $(0,18)$, $(0,36)$, $(1,13)$, $(1,33)$, $(2,17)$, $(2,24)$, $(3,12)$, $(3,38)$, $(4,25)$, $(4,28)$, $(5,45)$, $(6,37)$, $(6,43)$, $(7,20)$, $(7,26)$, $(8,15)$, $(8,16)$, $(9,39)$, $(10,44)$, $(11,44)$, $(12,1)$, $(13,39)$, $(14,42)$, $(15,42)$, $(16,6)$, $(17,46)$, $(18,41)$, $(19,2)$, $(20,0)$, $(21,7)$, $(22,47)$, $(23,7)$, $(24,40)$, $(25,0)$, $(26,40)$, $(27,47)$, $(28,45)$, $(29,6)$, $(30,1)$, $(31,41)$, $(32,5)$, $(33,5)$, $(34,8)$, $(35,46)$, $(36,3)$, $(37,3)$, $(38,8)$, $(39,31)$, $(39,32)$, $(40,21)$, $(40,27)$, $(41,4)$, $(41,10)$, $(42,2)$, $(43,19)$, $(43,22)$, $(44,9)$, $(44,35)$, $(45,23)$, $(45,30)$, $(46,14)$, $(46,34)$, $(47,11)$, $(47,29)$ $\}$.

$C_{63} = \{$ $(0,43)$, $(0,47)$, $(1,40)$, $(1,41)$, $(2,13)$, $(2,52)$, $(3,0)$, $(3,36)$, $(4,45)$, $(5,11)$, $(5,25)$, $(6,28)$, $(7,5)$, $(8,44)$, $(9,23)$, $(9,35)$, $(10,18)$, $(11,56)$, $(12,7)$, $(13,12)$, $(13,32)$, $(14,45)$, $(15,3)$, $(16,34)$, $(17,46)$, $(17,61)$, $(18,42)$, $(18,53)$, $(19,4)$, $(20,4)$, $(21,8)$, $(22,54)$, $(23,14)$, $(23,33)$, $(24,7)$, $(25,2)$, $(26,60)$, $(27,31)$, $(28,56)$, $(29,59)$, $(30,5)$, $(31,24)$, $(31,38)$, $(32,57)$, $(33,3)$, $(34,6)$, $(35,31)$, $(36,2)$, $(37,60)$, $(38,55)$, $(39,29)$, $(39,48)$, $(40,8)$, $(41,54)$, $(42,58)$, $(43,58)$, $(44,9)$, $(44,20)$, $(45,1)$, $(45,16)$, $(46,28)$, $(47,59)$, $(48,17)$, $(49,30)$, $(49,50)$, $(50,55)$, $(51,6)$, $(52,44)$, $(53,27)$, $(53,39)$, $(54,18)$, $(55,57)$, $(56,34)$, $(57,37)$, $(57,51)$, $(58,17)$, $(59,26)$, $(59,62)$, $(60,10)$, $(60,49)$, $(61,21)$, $(61,22)$, $(62,15)$, $(62,19)$ $\}$.

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